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Connection Considerations of Gravitational Field in Finsler Spaces

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Some alternative connection structures of the Finslerian gravitational field are considered by modifying the independent variables (x, y) (x: point and y: vector) in various ways. For example, (x^k, y^i) $(k, i = 1, 2, 3, 4)$ are changed to (x^k, y^0) (y^0: scalar) or (x^0, y^i) (x^0: time axis); (x^k, y^i) are generalized to (x^k, y^i, p_i) (p_i: covector dual to y^i) or (x^k, y^i, q_a) (q_a: covector different from p_i); (x^k, y^i) are further generalized to $(x^k, y^{(a)i})$ $(a = 1, 2, \ldots, m)$, ($y^{(a)}$: (a)th vector), etc.

KEY WORDS: Finsler geometry; Finslerian relativity; connections; gravitational field.

1. INTRODUCTION

A peculiar velocity field is produced by the gravity of mass fluctuations which are due to the anisotropic distribution of particles. The nonhomogeneity and anisotropy of the gravitational field cause our motion and the material contents of the universe. That means that the anisotropic and nonhomogeneous field intrinsically include the motions. Therefore, a standard interpretation for the cosmic background radiation dipole anisotropy is the result of our peculiar motion caused by the gravitational field of the irregularities in the mass distribution (Peebles, 1993).

For the above mentioned reasons, a geometrical model that fulfills the assumptions of a geometrical interpretation of anisotropic distribution of matter for the gravitational field is the Finsler geometry. (Asanov, 1985; Ikeda, 1995; Stavrinos and Diakogiannis, 2004; Stavrinos, 2005; Vacaru, 1997).

In the theory of gravitational field in Finsler spaces, the independent variables are chosen as (x^k, y^i), $(k, i = 1, 2, 3, 4)$, where the vector y is attached to each

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point x as the internal variable. Therefore, the Finslerian gravitational field is regarded as a unified field between the external (x)-field spanned by points x and the internal y-field spanned by vectors y.

From the vector bundle-like standpoint (Miron and Anastasiei, 1997), the Finslerian gravitational field is considered the unified field over the total space of the vector bundle whose base manifold is the (x)-field and fiber at each point x is the (y)-field. That is to say, the Finslerian gravitational field can be treated by means of differential geometry of the total space of the vector bundle mentioned above.

Along this line, if the independent variables (x, y) are generalized or specialized in various ways, then the so-called adapted frame is also changed accordingly. By doing so, we can consider some interesting modified connection structures associated with the Finslerian gravitational field. In the following section, we shall show some concrete examples: (x^k, y^i) are changed to (x^k, y^0), y^0: scalar or (x^0, y^i) (x^0: time axis); (x^k, y^i) are generalized to (x^k, y^i, p_i) (p_i: covector dual to y^i) or (x^k, y^i, q_a) (q_a: covector different from p_i); (x^k, y^i) are further generalized to $(x^k, y^{(a)i})$ $(a = 1, 2, \ldots, m)$, ($y^{(a)}$: (a)-th vector), etc.

2. FINSLERIAN CONNECTION STRUCTURE

In the total space, the adapted frame is set as follows (Miron and Anastasiei, 1997):

$$\begin{cases} dx^A \equiv \left(dx^k, \delta y^i = dy^i + N^i_\lambda dx^\lambda\right) \\ \frac{\partial}{\partial x^A} \equiv \left(\frac{\delta}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} - N^i_\lambda \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^i}\right) \end{cases} \tag{1}$$

where N^i_λ denotes the nonlinear connection representing physically the interaction between the (x)- and (y)-fields, and $X^A = (x^k, y^i)$ $(A = (k, i) = 1, 2, \ldots, 8)$. The intrinsic connection δy represents the intrinsic behavior of the internal variable y: For example, if the intrinsic behavior of y is grasped by $\bar{y}^i = K^i_j(x) y^j$, $K^i_j(x)$ being the rotation matrix, then δy is given by (Ikeda, 2000)

$$\begin{aligned} \delta y^i &= dy^i + N^i_\lambda dx^\lambda (= 0) \\ dy^i &\equiv \bar{y}^i - K^i_j(0) y^j \\ N^i_\lambda &\equiv -\frac{\partial K^i_j(x)}{\partial x^\lambda} y^j \end{aligned} \tag{2}$$

where we have put $K^i_j(x) = K^i_j(0) + \frac{\partial K^i_j(x)}{\partial x^\lambda} dx^\lambda$.

By a physical point of view we may apply the above-mentioned form of nonlinear connection in the framework of the observed anisotropy of cosmic background radiation (CBR). As it has been studied in a Finslerian ansantz, it can be represented by a vector $\hat{\ell}^i(x)$, which is incorporated in a metric model of

Finsler geometry as a result of our motion with respect to some local frame in the universe (cf. Stavrinos and Diakogiannis, 2004; Stavrinos, 2005). The Lagrangian function of this metric is given by

$$\mathcal{L}(x, y) = \sqrt{g_{ij} y^i y^j} + \phi(x)\ell_a y^a \tag{3}$$

The rotation of a vector ℓ'^{α} from anisotropy axis ℓ^{β} is given in virtue of the rotation group $\Lambda^{\alpha}_{\beta}(x)$. In this case, the nonlinear connection

$$N^i_{\lambda} = -\frac{\partial \Lambda^i_{\beta}(x)}{\partial x^{\lambda}} \ell^{\beta} \tag{4}$$

expresses the variation of the rotation group with respect to the anisotropy axis. In Stavrinos and Diakogiannis (2004), the expression $\phi(x)\ell^{\alpha}$ represented the spin density, where the function $\phi(x)$ is related to the mass density, i.e. it depends on the angular velocity and mass distribution. The rotation group can be used instead of the function $\phi(x)$ in the right-hand side of (3), giving a profound geometrical and physical meaning to the concept of nonlinear connection, N, and to that of anisotropic behavior of the gravitational field of anisotropy axis. Then, if we choose $N^i_{\lambda} = \frac{\partial \phi(x)}{\partial x^{\lambda}} \ell^i$, we can see that the nonlinear connection expresses the spin density.

In addition, the Berwald type form of the nonlinear connection (4) is given by

$$N^i_{\lambda\kappa} = \frac{\partial N^i_{\lambda}}{\partial \ell^{\kappa}} = -\frac{\partial \Lambda^i_{\kappa}(x)}{\partial x^{\lambda}} \tag{5}$$

The tensor field

$$\Lambda^i_{\kappa\lambda} = N^i_{\kappa\lambda} - N^i_{\lambda\kappa} = \frac{\partial \Lambda^i_{\kappa}}{\partial x^{\lambda}} - \frac{\partial \Lambda^i_{\lambda}}{\partial x^{\kappa}} \tag{6}$$

denotes a torsion of variation of the rotation group Λ^i_{κ}. The form (6) denotes the curvature of nonlinear connection. The integrability conditions by (6) are defined by $\Lambda^i_{\kappa\lambda} = 0$.

The adapted frame (1) can be modified in the form, e.g.,

$$\begin{cases} dx^A \equiv \left(\delta x^k = dx^k + \Pi^k_i \delta y^i,\ \delta y^i = dy^i + N^i_{\lambda} dx^{\lambda}\right) \\ \frac{\partial}{\partial x^A} \equiv \left(\frac{\delta}{\delta x^{\lambda}} = \frac{\partial}{\partial x^{\lambda}} - N^i_{\lambda} \frac{\partial}{\partial y^i},\ \frac{\delta}{\delta y^i} = \frac{\partial}{\partial y^i} - \Pi^{\lambda}_i \frac{\delta}{\delta x^{\lambda}}\right) \end{cases} \tag{7}$$

etc., This case corresponds to the case of contact transformation (Miron *et al.*, 2001).

On the basis of the adapted frame (1), the Finslerian connection structure is introduced by Miron and Anastasiei (1997)

$$\nabla_{\frac{\partial}{\partial x^C}} \frac{\partial}{\partial x^B} = \Gamma^A_{BC} \frac{\partial}{\partial x^A}$$

$$\Gamma^A_{BC} = \left(L^k_{\lambda\mu}, L^i_{j\mu}, C^k_{\lambda l}, C^i_{jl}\right) \tag{8}$$

Namely, four connection coefficients appear. Then, the following four kinds of covariant derivatives can be defined, for an arbitrary vector $V^A = (V^k, V^i)$:

$$\begin{cases} V^\kappa_{|\mu} &= \frac{\delta V^\kappa}{\delta x^\mu} + L^\kappa_{\lambda\mu} V^\lambda \\ V^\kappa|_l &= \frac{\partial V^\kappa}{\partial y^l} + C^\kappa_{\lambda l} V^\lambda \\ V^i_{|\mu} &= \frac{\delta V^i}{\delta x^\mu} + L^i_{j\mu} V^j \\ V^i|_l &= \frac{\delta V^i}{\delta y^l} + C^i_{jl} V^j \end{cases} \tag{9}$$

Further, the Finslerian metrical structure is given by (Miron and Anastasiei, 1997) $G \equiv G_{AB} dx^A dx^B = g_{\lambda\kappa} dx^\lambda \otimes dx^\kappa + g_{ij} dx^i \otimes dx^j$. The metrical conditions $g_{\lambda\kappa|\mu} = 0$, $g_{\lambda\kappa}|_l = 0$, $g_{ij|\mu} = 0$ and $g_{ij}|_l = 0$ can be imposed, if necessary.

3. MODIFIED CONNECTION STRUCTURES-I

We shall first consider the case where (x^k, y^i) are changed to (x^k, y^0), y^0 being a scalar. Then, the adapted frame is reduced to

$$\begin{cases} dx^A \equiv \left(dx^k, \delta y^0 = dy^0 + N^0_\lambda dx^\lambda\right) \\ \frac{\partial}{\partial x^A} \equiv \left(\frac{\delta}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} - N^0_\lambda \frac{\partial}{\partial y^0}, \frac{\partial}{\partial y^0}\right) \end{cases} \tag{10}$$

In this case, the connection structure and the metrical structure are given by

$$\Gamma^A_{BC} = (L^k_{\lambda\mu}, L^0_{0\mu}, C^k_{\lambda 0}, C^0_{00}) \tag{11}$$

and

$$G = g_{\lambda k} dx^k \otimes dx^\lambda + g_{00} \delta y^0 \otimes \delta y^0, \tag{12}$$

respectively.

In virtue of (2), if we choose the nonlinear connection in the form $N^0_\lambda = -\frac{\partial \Lambda^0_0}{\partial x^\lambda} y^0$ with $\Lambda^0_0(x) = \phi(x)$ and $y^0 = a(t)$, then we get

$$N^0_\lambda = -\frac{\partial \phi(x)}{\partial x^\lambda} y^0 = -\phi_{,\lambda}(x) a(t)$$

that means that N^0_λ takes the form of a covector caused by the fluctuations of material fields of the Finslerian gravitational field, $a(t)$ can be represented as a scale factor.

If y^0 is constant, then we can put in (10) $dy^0 \equiv 0$, $\delta y^0 \equiv N^0_\lambda dx^\lambda$ and $\frac{\delta}{\delta x^\lambda} \equiv \frac{\partial}{\partial x^\lambda}$, $\frac{\partial}{\partial y^0} \equiv 0$. Therefore we can put in (11) $\Gamma^A_{BC} = (L^k_{\lambda\mu}, L^0_{0\mu}, C^k_{\lambda 0} \equiv 0, C^0_{00} \equiv 0)$. $g_{00}(x^k, y^0)$ in (12) is not a constant, in general. If g_{00} is a constant, then we can put $L^0_{0\mu} \equiv 0$).

Next, we shall take up the case where (x^k, y^i) are changed to (x^0, y^i), x^0 being the time axis. This case is dual to the above mentioned case and means the time-sequence of the (y)-fields. In this case, the adapted frame becomes

$$\begin{cases} dx^A \equiv \left(dx^0, \delta y^i = dy^i + N_0^i dx^0\right) \\ \frac{\partial}{\partial x^A} \equiv \left(\frac{\delta}{\delta x^0} = \frac{\partial}{\partial x^0} - N_0^i \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^i}\right). \end{cases} \tag{13}$$

The nonlinear connection N_0^i plays the role similar to the shift function in the theory of general relativity (Bergman, 1961). The connection structure and the metrical structure are given by, respectively,

$$\Gamma_{BC}^A = \left(L_{0\mu}^0, L_{j0}^i, C_{0k}^0, C_{jk}^i\right) \tag{14}$$

and

$$G = g_{00} dx^0 \otimes dx^0 + g_{ij} \delta y^i \otimes \delta y^j. \tag{15}$$

4. MODIFIED CONNECTION STRUCTURES-II

In this section, we shall first take account of the case where the independent variables (x^k, y^i) are generalized to (x^k, y^i, p_i), p_i being a covector dual to y^i. Then, the adapted frame is set as follows: (Miron *et al.*, 2001):

$$\begin{cases} dx^A \equiv \left(dx^k, \delta y^i = dy^i + N_\lambda^i dx^\lambda, \delta p_j = dp_j - M_{\lambda j} dx^\lambda\right) \\ \frac{\partial}{\partial x^A} \equiv \left(\frac{\delta}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} - N_\lambda^i \frac{\partial}{\partial y^i} + M_{\lambda j} \frac{\partial}{\partial p_j}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial p_j}\right) \end{cases} \tag{16}$$

where two kinds of nonlinear connections N_λ^i and M_{ij} must be introduced. Then, the connection structure is given by

$$\Gamma_{BC}^A = \left(L_{\lambda\mu}^k, L_{j\mu}^i, C_{\lambda l}^\kappa, C_{jl}^i, E_\lambda^{\kappa l}, E_j^{il}\right) \tag{17}$$

also the following covariant derivatives can be defined (Miron *et al.*, 2001): For example,

$$\begin{cases} V^\kappa\ |_\mu = \frac{\delta V^k}{\delta x^\mu} + L_{\lambda\mu}^\kappa V^\lambda \\ V^i\ |_\kappa = \frac{\partial V^i}{\partial y^\kappa} + C_{j\kappa}^i V^j \\ V^\kappa\ |_l = \frac{\partial V^\kappa}{\partial P_l} + E_\lambda^{\kappa l} V^\lambda \end{cases} \tag{18}$$

etc. The metrical structure is introduced by

$$G = g_{\lambda\kappa} dx^\kappa \otimes dx^\lambda + g_{ij} \delta y^i \otimes \delta y^j + g^{ij} \delta p_i \otimes \delta p_j. \tag{19}$$

Next we shall consider the case where the independent variables become (x^k, y^i, q_a), q_a being a covector taken at one more microscopic level than the

p_i-level. Then, the spatial structure becomes second-order vector bundle-like and the adapted frame is set as follows:

$$\begin{cases} dx^A \equiv \left(dx^k, \delta y^i = dy^i + N^i_\lambda dx^\lambda, \delta q_a = dq_a - P_{\lambda\kappa} dx^\lambda - Q_{ia} dy^i\right) \\ \frac{\partial}{\partial x^A} \equiv \left(\frac{\delta}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} - N^i_\lambda \frac{\partial}{\partial y^i} + P_{\lambda a} \frac{\partial}{\partial q_a}, \frac{\delta}{\delta x^i} = \frac{\partial}{\partial y^i} + Q_{ia} \frac{\partial}{\partial q_a}, \frac{\partial}{\partial q_a}\right) \end{cases} \tag{20}$$

where three kinds of nonlinear connections N^i_λ, $P_{\lambda a}$, and Q_{ia} must be introduced. Therefore, the connection and metrical structures are given by,

$$\Gamma^A_{BC} = \left(L^k_{\lambda\mu}, L^i_{j\mu}, L^a_{b\mu}, C^k_{\lambda l}, C^i_{jl}, C^a_{bl}, H^{kc}_\lambda, H^{ic}_j, H^{ac}_b\right) \tag{21}$$

and

$$G = g_{\lambda k} dx^k \otimes dx^\lambda + g_{ij} \delta y^i \otimes \delta y^j + g^{ab} \delta q_a \otimes \delta q_b, \text{ respectively.} \tag{22}$$

From (21), for example, the following covariant derivatives can be defined:

$$\begin{cases} V^\kappa_{|\mu} = \frac{\delta V^\kappa}{\delta x^\mu} + L^\kappa_{\lambda\mu} V^\lambda \\ V^i\ |_l = \frac{\delta V^i}{\delta y^l} + C^i_{jl} V^j \\ V^\kappa\ |_c = \frac{\partial V^\kappa}{\partial q_c} + H^{\kappa c}_\lambda V^\lambda \end{cases} \tag{23}$$

etc.

5. MODIFIED CONNECTION STRUCTURES-III

Finally, we shall consider the case where (x^k, y^i) are further generalized to $(x^k, y^{(\alpha)i})$ $(\alpha = 1, 2, \ldots, m)$, $y^{(\alpha)}$ being the (α)-th vector interacting physically with $y^{(\beta)}$ $(\alpha \neq \beta)$. Then, the adapted frame is set as follows:

$$\begin{cases} dx^A \equiv \left(dx^k, \delta y^{(\alpha)i} = N^{(\alpha)i}_\lambda dx^\lambda + \sum\limits_{\beta=1}^{\alpha} \Psi^{(\alpha)i}_{(\beta)k} dy^{(\beta)k}\right) \\ \frac{\partial}{\partial x^A} \equiv \left(\frac{\delta}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} - \sum\limits_{\beta=1}^{m} N^{(\alpha)i}_\lambda \frac{\delta}{\delta y^{(\beta)i}}, \frac{\delta}{\delta y^{(\alpha)i}} = \sum\limits_{\beta=\alpha}^{m} (\Psi^{-1})^{(\beta)j}_{(\alpha)i} \frac{\partial}{\partial y^{(\beta)j}}\right) \end{cases} \tag{24}$$

where $\delta y^{(\alpha)}$ is written in this form in comparison with the base connection in the theory of higher order spaces (cf. Kawaguchi, 1932). The quantity $N^{(\alpha)i}_\lambda$ denotes the generalized nonlinear connection and $\Psi^{(\alpha)i}_{(\beta)j}$ means the interaction of $y^{(\alpha)}$ and $y^{(\beta)}$. Then, the connection structure becomes

$$\Gamma^A_{BC} = \left(L^k_{\lambda\mu}, L^{(\alpha)i}_{(\beta)j\mu}, C^k_{\lambda(\gamma)k}, C^{(\alpha)i}_{(\beta)j(\gamma)k}\right) \tag{25}$$

and the metrical structure is given by

$$G = g_{\lambda k} dx^k \otimes dx^\lambda + \sum_{\alpha=1}^{m} g_{(\alpha)i(\beta)j} \delta y^{(\alpha)i} \otimes \delta y^{(\beta)j} \tag{26}$$

In (25) and (26), $L^{(\alpha)i}_{(\beta)j\mu}$ and $C^{(\alpha)i}_{(\beta)j(\gamma)k}$ can be reduced to $L^{(\alpha)i}_{(\alpha)j\mu}$ and $C^{(\alpha)i}_{(\alpha)j(\gamma)k}$ and $g_{(\alpha)i(\beta)j}$ can be reduced to $g_{(\alpha)i(\alpha)j}$, under some convenient conditions (Miron, 1997). Thus it is understood from the above that we can consider many interesting modified connection structures by generalizing the independent variables (or the adapted frame) in various ways.

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